5 DEFLECTIONS

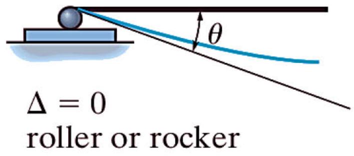
5.1 Deflection Diagrams and the Elastic Curve

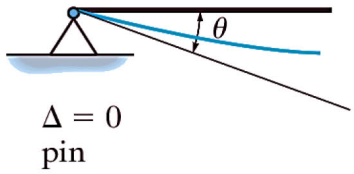
Deflections of structures can occur from various sources, such as loads, temperature, fabrication errors, or settlement. In design, deflections must be limited in order to provide integrity and stability of roofs, and prevent cracking of attached brittle materials such as concrete, plaster or glass.

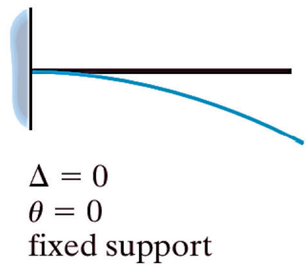
Furthermore, a structure must not vibrate or deflect severely in order to “appear” safe for its occupants. More important, though, deflections at specified points in a structure must be determined if one is to analyze statically indeterminate structures.

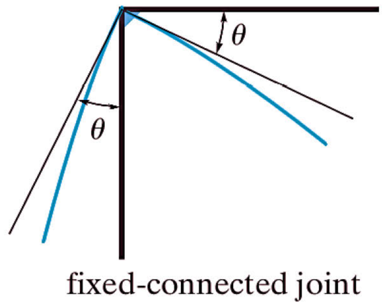
The deflections to be considered throughout this text apply only to structures having ***linear elastic material response***. Under this condition, a structure subjected to a load will return to its original undeformed position after the load is removed.

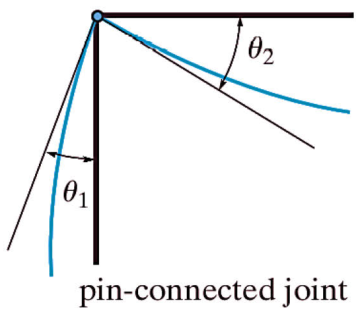
- ✓ The deflection of a structure is caused by its internal loadings such as normal force, shear force, or bending moment.
- ✓ For ***beams*** and ***frames***, however, the greatest deflections are most often caused by ***internal bending***, whereas ***internal axial forces*** cause the deflections of a ***truss***.
- ✓ Before the slope or displacement of a point on a beam or frame is determined, it is often helpful to sketch the deflected shape of the structure when it is loaded in order to partially check the results.
- ✓ This ***deflection diagram*** represents the ***elastic curve*** or locus of points which defines the displaced position of the centroid of the cross section along the members.
- ✓ If the elastic curve seems difficult to establish, it is suggested that the moment diagram for the beam or frame be drawn first.
- ✓ A ***positive*** moment tends to bend a beam or horizontal member ***concave upward***. Likewise, a ***negative moment*** tends to bend the beam or member ***concave downward***,

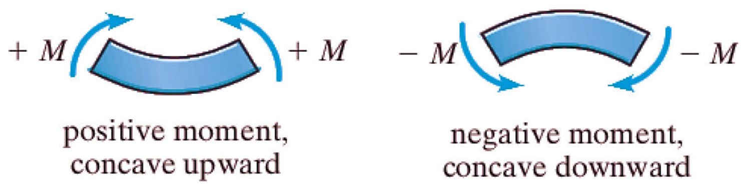




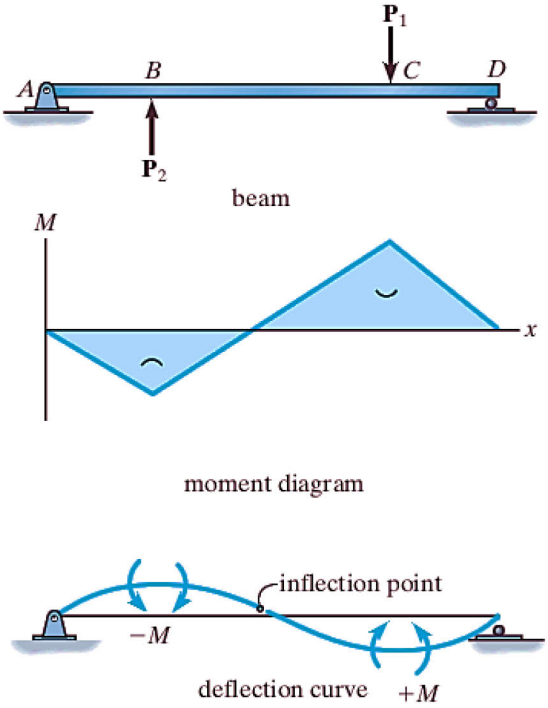


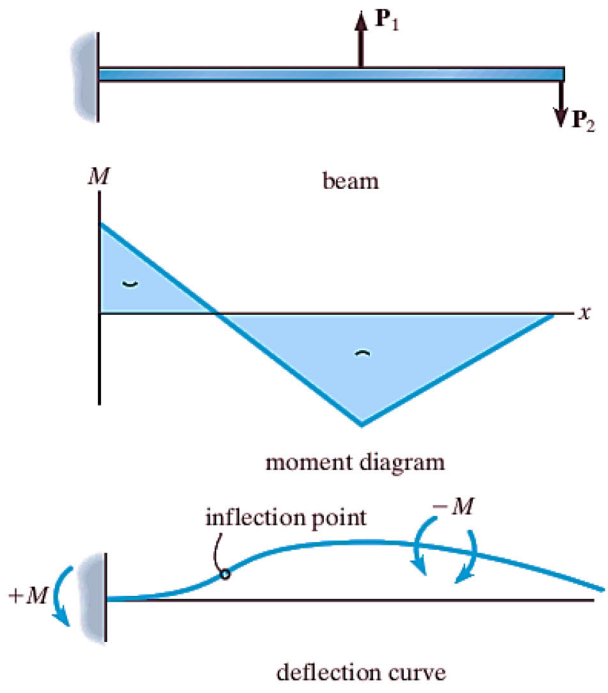






Deflection Diagrams and the Elastic Curve: The Double Integration Method

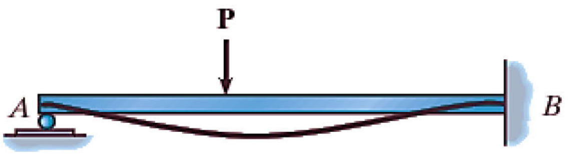




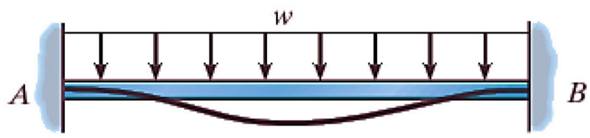
EXAMPLE 5.1.1

Draw the deflected shape of each of the beams shown in Figures.

Solution



(a)



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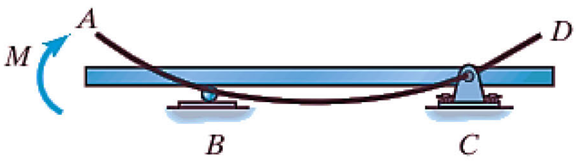
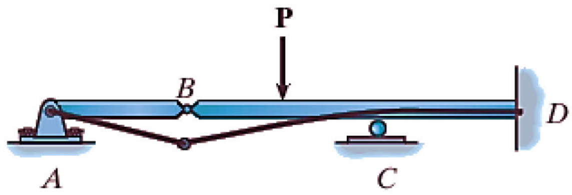
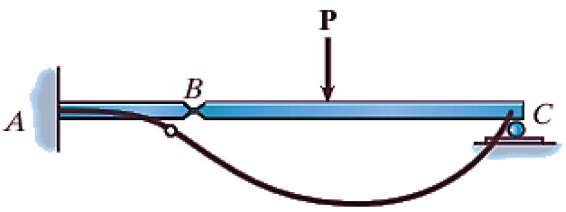


(c)



(d)



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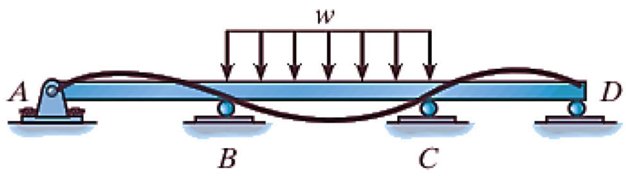
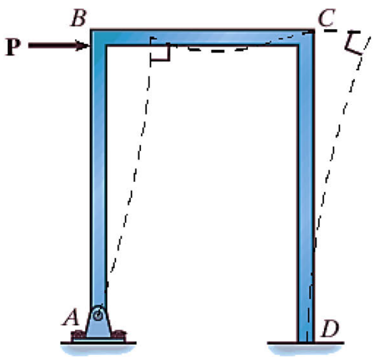


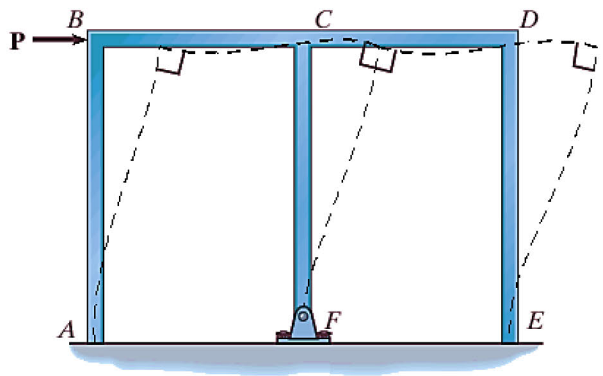
(f)

EXAMPLE 5.1.2
Draw the deflected shape of each of the frames shown in Figures.

Solution



(a)



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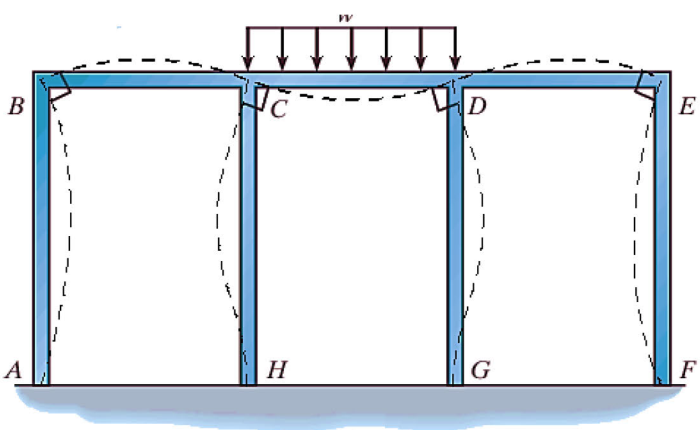

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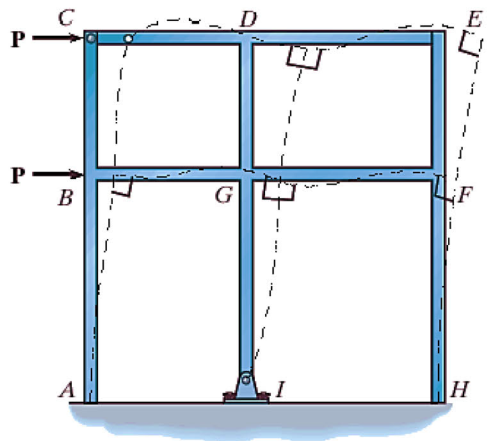
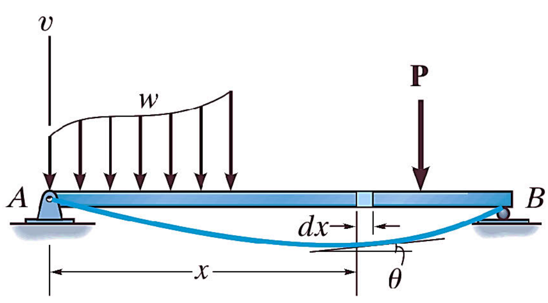


(d)

5.2 Elastic-Beam Theory

When the internal moment ***M*** deforms the element of the beam, each cross section remains plane and the angle between them becomes ***dθ***, **Fig. *b***. The arc ***dx*** that represents a portion of the elastic curve intersects the neutral axis for each cross section. The ***radius of curvature*** for this arc is defined as the distance , which is measured from ***the center of curvature O′*** to ***dx***. Any arc on the element other than ***dx*** is subjected to a normal strain.



(a)

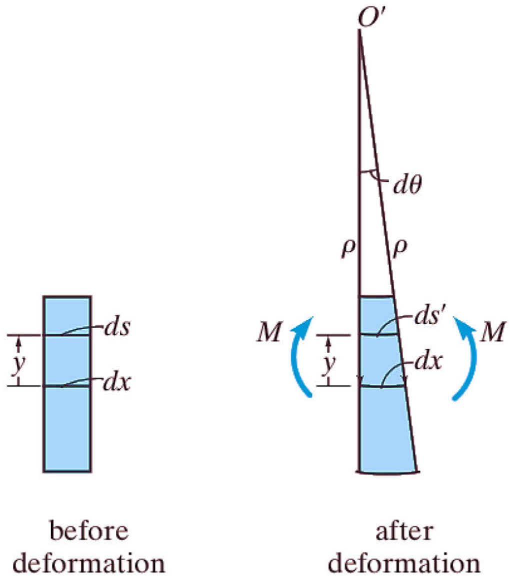
For example, the strain in arc ***ds***, located at a position ***y*** from the neutral axis, is

$$\epsilon = (ds' - ds)/ds$$

However,

$ds = dx = \rho d\theta$ and $ds' = (\rho - y)/d\theta$ and so

$$\epsilon = \frac{(\rho - y)d\theta - \rho d\theta}{\rho d\theta} \quad \text{or} \quad \frac{1}{\rho} = \frac{\epsilon}{y}$$



(b)

If the material is homogeneous and behaves in a linear elastic manner, then Hooke's law applies, $\epsilon = \frac{\sigma}{E}$.

Also, since the flexure formula applies, $\sigma = -\frac{My}{I}$.

Combining these equations and substituting into the above equation, we have

$$\frac{1}{\rho} = \frac{M}{EI} \qquad ...(1)$$

Here

ρ = the radius of curvature at a specific point on the elastic curve (**1/ *ρ*** is referred to as the ***curvature***)
M = the internal moment in the beam at the point where ***ρ*** is to be determined
E = the material's modulus of elasticity
I = the beam's moment of inertia computed about the neutral axis

The product ***EI*** in this equation is referred to as the ***flexural rigidity***, and it is always a positive quantity. Since ***dx* = *ρ dθ***, then from **Eq. 1**

$$d\theta = \frac{M}{EI}dx \qquad ...(2)$$

If we choose the ***v*** axis positive upward, **Fig. *a***, and if we can express the curvature **(1/*ρ*)** in terms of ***x*** and ***v***, we can then determine the elastic curve for the beam. The curvature relationship is

$$\frac{1}{\rho} = \frac{d^2v/dx^2}{\left[1 + (dv/dx)^2\right]^{3/2}}$$

Therefore,

$$\frac{M}{EI} = \frac{d^2v/dx^2}{\left[1 + (dv/dx)^2\right]^{3/2}} \qquad ...(3)$$

This equation represents a nonlinear second-order differential equation. Its solution, ***v* = *f(x)***, gives the exact shape of the elastic curve-assuming, of course, that beam deflections occur only due to bending. In order to facilitate the solution of a greater number of problems, **Eq. 3** will be modified by making an important simplification. Since the slope of the elastic curve for most structures is very small, we will use <u>small deflection theory</u> and assume ***dv/dx*≈ 0**.

Consequently its square will be negligible compared to unity and therefore **Eq. 3** reduces to

$$\frac{d^2v}{dx^2} = \frac{M}{EI} \qquad ...(4)$$

It should also be pointed out that by assuming ***dv/dx*≈ 0,** the original length of the beam's axis ***x*** and the ***arc*** of its elastic curve will be approximately the same. In other words, ***ds*** in **Fig. *b*** is approximately equal to ***dx***, since

$$ds = \sqrt{dx^2 + dv^2} = \sqrt{1 + (dv/dx)^2}\,dx \approx dx$$

This result implies that points on the elastic curve will only be displaced vertically and not horizontally.

5.3 The Double Integration Method

Once ***M*** is expressed as a function of position ***x***, then successive integrations of **Eq. 4** will yield the beam's slope, $\boldsymbol{\theta \approx \tan\theta = dv/dx = \int (M/EI)\,dx}$ and the equation of the elastic curve, $\boldsymbol{v = f(x) = \iint (M/EI)\,dx}$ respectively.

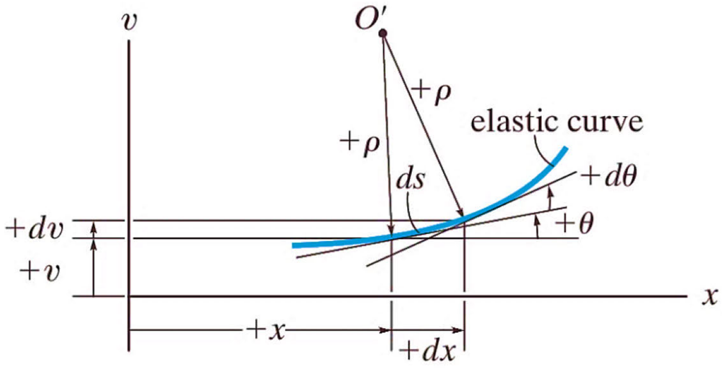
For each integration it is necessary to introduce a "constant of integration" and then solve for the constants to obtain a unique solution for a particular problem.

<u>Sign Convention</u>

When applying **Eq.4**, it is important to use the proper sign for ***M*** as established by the sign convention that was used in the derivation of this equation, **Fig.*a***.

Furthermore, recall that positive deflection, ***v***, is upward, and as a result, the positive slope angle ***θ*** will be measured counterclockwise from the *x* axis. The reason for this is shown in **Fig.*b***. Here, positive increases ***dx*** and ***d*** in ***x*** and create an increase ***dθ*** that is counterclockwise. Also, since the slope angle will be very small, its value in radians can be determined directly from ***θ*** **≈ tan *θ* = *dv/dx***

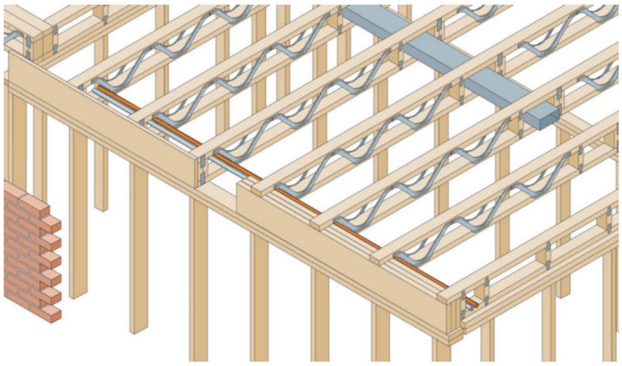
(a)



(b)

EXAMPLE 5.3.1

Each simply supported floor joist shown in the photo is subjected to a uniform design loading of **4 kN/m**, **Fig.*a***. Determine the maximum deflection of the joist. ***EI*** is constant.

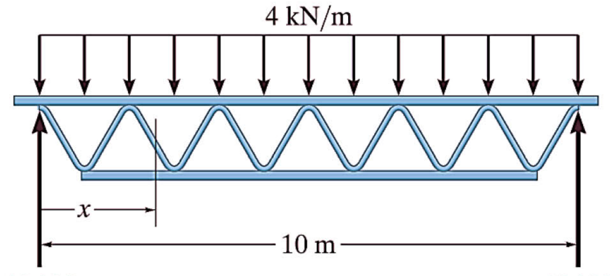


Solution

Elastic Curve.
Due to symmetry, the joist's maximum deflection will occur at its center.

Moment Function.
From the free-body diagram, **Fig.*b***, we have

$$M = 20x - 4x\left(\frac{x}{2}\right) = 20x - 2x^2$$



(a)

Slope and Elastic Curve.
Applying **Eq. 4** and integrating twice gives

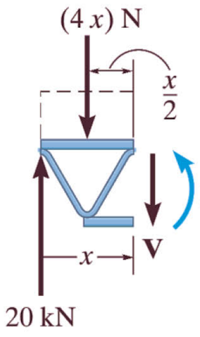
$$EI\frac{d^2v}{dx^2} = 20x - 2x^2$$

$$EI\frac{dv}{dx} = 10x^2 - 0.6667x^3 + C_1$$

$$EIv = 3.333x^3 - 0.1667x^4 + C_1x + C_2$$

Here
$v = 0$ at $x = 0$ so that $C_2 = 0$,
and
$v = 0$ at $x = 10$, so that $C_1 = -166.7$.
The equation of the elastic curve is therefore

$$EIv = 3.333x^3 - 0.1667x^4 - 1.667x$$



(b)

At $x = 5$ **m** note that $dv/dx = 0$ The maximum deflection is therefore

$$v_{max} = -\frac{521}{EI}$$

EXAMPLE 5.3.2

The cantilevered beam shown in **Fig.*a*** is subjected to a couple moment M_0 at its end. Determine the equation of the elastic curve. ***EI*** is constant.

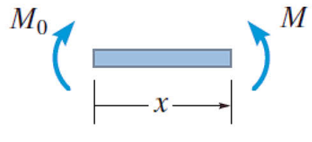
(a)

Solution

Elastic Curve.
The load tends to deflect the beam as shown in **Fig.*a*.** By inspection, the internal moment can be represented throughout the beam.

Moment Function.
From the free-body diagram, with M acting in the ***positive direction***, **Fig.b** we have

$$M = M_0$$



(b)

Slope and Elastic Curve.
Applying **Eq.4** and integrating twice yields

$$EI\frac{d^2v}{dx^2} = M_0$$

$$EI\frac{dv}{dx} = M_0x + C_1$$

$$EIv = \frac{M_0x^2}{2} + C_1x + C_2$$

Using the boundary conditions a $dv/dx = 0$ at $x = 0$ and $v = 0$ at $x = 0$, then $C_1 = C_2 = 0$.

$$\theta = \frac{dv}{dx} = \frac{M_0x}{EI}$$

$$v = \frac{M_0x^2}{2EI}$$

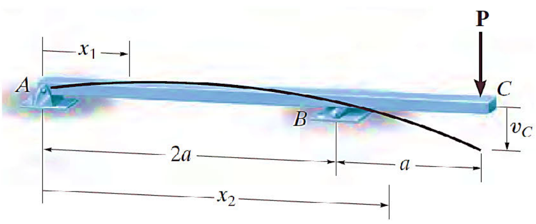
Maximum slope and displacement occur at **A ($x = L$)**, for which

$$\theta_A = \frac{M_0L}{EI}$$

$$v_A = \frac{M_0L^2}{2EI}$$

The ***positive*** result for θ_A indicates ***counterclockwise*** rotation and the ***positive*** result for v_A indicates that v_A is ***upward***. This agrees with the results sketched in **Fig. *a***.

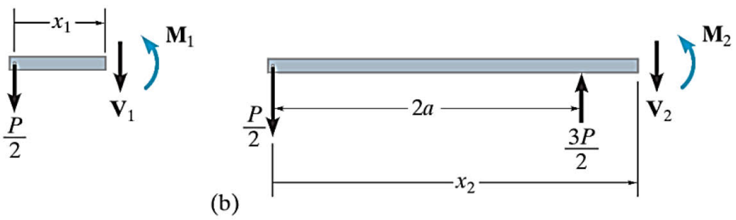
EXAMPLE 5.3.3
The beam in **Fig.*a*** is subjected to a load **P** at its end. Determine the displacement at C. EI is constant.



(a)

Solution 1
Elastic Curve.
The beam deflects into the shape shown in **Fig.*a***. Due to the loading, two x coordinates must be considered.

Moment Functions.
Using the free-body diagrams shown in **Fig.*b***, we have



(b)

$$M_1 = -\frac{P}{2}x_1 \qquad 0 \le x_1 \le 2a$$

$$M_2 = -\frac{P}{2}x_2 + \frac{3P}{2}(x_2 - 2a)$$

$$= Px_2 - 3Pa \qquad 2a \le x_1 \le 3a$$

Slope and Elastic Curve.

Applying **Eq.4**,

for x_1

$$EI\frac{d^2v_1}{dx_1^2} = -\frac{P}{2}x_1$$

$$EI\frac{dv_1}{dx_1} = -\frac{P}{4}x_1^2 + C_1 \qquad (1)$$

$$EIv_1 = -\frac{P}{12}x_1^3 + C_1x_1 + C_2 \qquad (2)$$

For x_2

$$EI\frac{d^2v_2}{dx_2^2} = Px_2 - 3Pa$$

$$EI\frac{dv_2}{dx_1} = \frac{P}{2}x_2^2 - 3Pax_2 + C_3 \qquad (3)$$

$$EIv_2 = \frac{P}{6}x_2^3 - \frac{3}{2}Pax_2^2 + C_3x_2 + C_4 \qquad (4)$$

The *four* constants of integration are determined using *three* boundary conditions, namely $\boldsymbol{v_1 = 0}$ at $\boldsymbol{x_1 = 0}$, $\boldsymbol{v_1 = 0}$ at $\boldsymbol{x_1 = 2a}$ and $\boldsymbol{v_2 = 0}$ at $\boldsymbol{x_2 = 2a}$, and *one* continuity equation.
Here the continuity of slope at the roller requires $\boldsymbol{dv_1/dx_1 = dv_2/dx_2}$ at $\boldsymbol{x_1 = x_2 = 2a}$.
Applying these four conditions yields

$$v_1 = 0 \text{ at } x_1 = 0; \qquad 0 = 0 + 0 + C_2$$

$$v_1 = 0 \text{ at } x_1 = 2a; \qquad 0 = -\frac{P}{12}(2a)^3 + C_1(2a) + C_2$$

$$v_2 = 0 \text{ at } x_2 = 2a; \qquad 0 = \frac{P}{6}(2a)^3 - \frac{3}{2}Pa(2a)^2 + C_3(2a) + C_4$$

$$\frac{dv_1(2\text{a})}{dx_1} = \frac{dv_2(2\text{a})}{dx_2}; \qquad -\frac{P}{4}(2a)^2 + C_1 = \frac{P}{2}(2a)^2 - 3Pa(2a) + \text{C}_3$$

Solving, we obtain

$$C_1 = \frac{Pa^2}{3} \qquad C_2 = 0 \qquad C_3 = \frac{10}{3}Pa^2 \qquad C_4 = -2Pa^3$$

Substituting C_3 and C_4 into Eq.(4) gives

$$v_2 = \frac{P}{6EI}x_2^3 - \frac{3Pa}{2EI}x_2^2 + \frac{10Pa^2}{3EI}x_2 - \frac{2Pa^3}{EI}$$

The displacement at ***C*** is determined by setting $\boldsymbol{x_2 = 3a}$ We get

$$v_C = -\frac{Pa^3}{EI}$$